Transformation matrix of a linear map

Two and a half weeks

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Module

This chapter is part of the ALM module: linear maps and matrices. Its is part of the B4 bimester.

The prerequisites are:

- Module [Maths-S1B1-OBM] from the seminar.
- Module [Maths-S2B3-B4-EV] about vector spaces.
- The previous chapter of this module, about linear maps.

Context

We have already mentioned in previous tutorials that linear algebra offers very useful models in the analysis of different computer engineering problems. This affirmation would not make sense if there was no way to store information held by a linear map in the machine. That is, to reduce to a finite size the whole information held by a linear map. In the general case, this size reduction is not possible, except in specific situations that you will soon come to study, such as signal mathematics.

However, if we restrict to finite dimensional vector spaces, this reduction is possible. In this case, we can store all the information held by a linear map in a table of values. The number of ways to do this reduction is the same as the number of bases for the input (domain) and output (codomain) spaces, i.e an infinite number. The purpose of this worksheet is to study these reductions.

Learning outcomes

At the end of this chapter, you are expected to be able to;

- Handle matrix operations (addition, multiplication, inverse).
- Switch from a linear map to its matrix representation and vice versa.
- Find the transition matrix from a basis to another, and use it.

More precisely, you should be able to:

- Add and multiply matrices.
- Find the inverse of a 3×3 matrix.
- Find the matrix of a linear map in a given pair of input/output bases.
- Use the matrix of a linear map to find the image of an input vector.
- Use the matrix of a linear map to analyze this mapping.
- Find the transition matrix from a basis to another.
- Use the transition matrix to connect the coordinates of a vector in different bases.

Convention 1. Let $n \in \mathbb{N}^*$. We denote by $e_1, \ldots, e_n$ the elements from the standard basis of $\mathbb{R}^n$. More specifically for $i \in \{1, \ldots, n\}$, e_i is the element whose components are all 0 except the i-th one which equals 1.

1 Characterizing a linear map by the image of a basis

1.1 Summary

Let E and F be two $\mathbb{R}$-vector spaces. To determine the image of ***any*** element of E by a linear map $f : E \to F$ it is necessary and sufficient to determine the image of ***one*** arbitrary basis from E. This section focuses on this property.

1.2 Exercises

Exercise 1.1
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:

$$\begin{aligned} f(e_1) &= (2, 1) \\ f(e_2) &= (-1, 1) \\ f(e_3) &= (1, 7). \end{aligned}$$

1. Can f map $(1, -1, 0)$ to $(3, 1)$?
2. What is the image of $(1, 2, 3)$ by f ?
3. More generally speaking what is the image by f of any vector $(x, y, z) \in \mathbb{R}^3$ as a function of its coordinates x, y and z?

Exercise 1.2
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map such that:

$$\begin{aligned} f\big((0, 1, 1)\big) &= (2, 1) \\ f\big((0, 1, 0)\big) &= (-1, 1) \\ f\big((0, -1, 1)\big) &= (4, -1). \end{aligned}$$

1. What is the image of $(0, 1, 2)$ by f?
2. What is the image of $(1, 0, 2)$ by f?
3. What is the problem?

Exercise 1.3
During one of your tutorial sessions, your teacher defines a linear map $f : \mathbb{R}^3$ to $\mathbb{R}^3$ by:

$$\begin{aligned} f\big((1, 2, 1)\big) &= (1, 1, 1) \\ f\big((-1, 1, 1)\big) &= (-1, 1, 1) \\ f\big((0, 3, 2)\big) &= (0, 0, 1) \end{aligned}$$

Why do you think he has done a *typing error*?

Exercise 1.4
1. Let E and F be two vector spaces over $\mathbb{R}$, f and g two linear maps from E to F. Assume furthermore that E is finite-dimensional and let $\mathcal{B}$ be a basis of E.
   (a) Express mathematically (with quantifiers) the property: $f = g$.
   (b) Show that: $(f = g) \iff (\forall u \in \mathcal{B}, \quad f(u) = g(u))$

2. Let F be a $\mathbb{R}$-vs, $(f,g) \in \left(\mathcal{L}(\mathbb{R}^2, F)\right)^2$ and A a subset of $\mathbb{R}^2$.

   Assume that $\forall (u,v) \in A^2$, $f(u) = g(u)$. Can we deduce that $f = g$ (on $\mathbb{R}^2$) in the following cases?

   (a) $A = \mathbb{N}^2$

   (b) A is the disc of radius 1, centered at the $0_{\mathbb{R}^2}$?

   (c) A is a vector line of $\mathbb{R}^2$?

   (d) A is the line of equation $y = 2x + 1$?

Exercise 1.5

1. Let $f \in \mathcal{L}(\mathbb{R}^2)$ such that $f(e_1) = (2,3)$ and $f(e_2) = (-1,5)$ where (e_1, e_2) is the standard basis of $\mathbb{R}^2$.

   (a) Let $u = (x,y) \in \mathbb{R}^2$. What is the value of $f(u)$?

   (b) Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ and $U = \begin{pmatrix} x \\ y \end{pmatrix}$. Compute AU. What do you notice?

2. Let $\varepsilon_1 = (1,-1)$ and $\varepsilon_2 = (1,0)$. Consider $f \in \mathcal{L}(\mathbb{R}^2)$ such that $f(\varepsilon 1) = (2,3)$ and $f(\varepsilon_2) = (-1,5)$.

   (a) Let $u = (x,y) \in \mathbb{R}^2$. Do the data above enable one to compute $f(u)$? In case of positive answer, do the computation.

   (b) Which matrix multiplication leads to this answer?

2 Matrix of a linear map in a given basis

2.1 Summary

The previous section led us to observe that the information held in a linear map f can be represented by the values of f at a basis of the input space. In the case where both the input and output spaces are finite dimensional, we can represent a linear map with a table of dimension two: if $\boldsymbol{B} = (b_i)_{i=1}^p$ and $\boldsymbol{C} = (c_j)_{j=1}^n$ are respectively bases of the input and output space, then the i-th column of the table stores the coordinates of $f(b_i)$ in the basis $\boldsymbol{C}$.

This table is called the ***matrix of f in the bases $\boldsymbol{B}$ and $\boldsymbol{C}$***. It contains all the information held by f, provided that we know what the bases $\boldsymbol{B}$ and $\boldsymbol{C}$ are. Changing one of these bases leads to another matrix to represent f.

Given these bases $\boldsymbol{B}$ and $\boldsymbol{C}$, associating a matrix to a linear map is not only a manner of *storing* the information held by f. We can actually leave f aside and only use the matrix. We could actually sum up most of the linear algebra issues to a single question: ***How to choose the right basis for the linear map we are analyzing?***

Hypothesis 2.1. We will only consider finite dimensional vector spaces.

2.2 Available MiMos

Mimo: Définition d'une matrice

Mimo: Multiplication de matrices

Mimo: Matrice d'une application linéaire

2.3 Exercises

Exercise 2.6
Compute the matrices of the following linear maps in the given bases:

1. $f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (2x+y, x-y) \end{cases}$

   (a) in the standard basis of $\mathbb{R}^2$ as input and output bases,
   (b) in the standard basis as input basis and $(\varepsilon_1{=}(1,1), \varepsilon_2{=}(1,-1))$ as output basis
   (c) in $(\varepsilon_1{=}(1,1), \varepsilon_2{=}(1,-1))$ as input basis and the standard basis as output,
   (d) in $(\varepsilon_2{=}(1,-1), \varepsilon_1{=}(1,1))$ as input basis and the standard basis as output,
   (e) in $(\varepsilon_1{=}(1,1), \varepsilon_2{=}(1,-1))$ as input basis and $((0,1),(1,1))$ as output basis.

2. $g : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (3x-y+z, y-2z) \end{cases}$

   (a) in the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$,
   (b) in $((0,1,1),(1,0,1),(1,1,0))$ as input basis and $((1,-1),(1,1))$ as output basis.

3. ***(Homework)*** $h : \begin{cases} \mathbb{R}^4 & \longrightarrow & \mathbb{R} \\ (x,y,z,t) & \longmapsto & x-2y+3t \end{cases}$

   (a) in the standard input and output bases
   (b) in the standard basis as input and (2) as output basis.

★ 4. $\ell : \begin{cases} \mathbb{R}^p & \longrightarrow & \mathbb{R}^n \\ \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} & \longmapsto & \begin{pmatrix} a_{11}x_1 + \cdots + a_{1p}x_p \\ \vdots \\ a_{n1}x_1 + \cdots + a_{np}x_p \end{pmatrix} \end{cases}$ where a_{ij} are real numbers.

   in the standard bases of $\mathbb{R}^p$ and $\mathbb{R}^n$.

5. $\phi : \begin{cases} \mathbb{R}_3[X] & \longrightarrow & \mathbb{R}_3[X] \\ P & \longmapsto & P' \end{cases}$.

   (a) in the standard basis $(1, X, X^2, X^3)$ as input and output bases,
   (b) in $(1, X+1, X^2-1, X^3+X)$ as input basis and the standard basis as output,
   (c) in the standard basis as input and $(1, X+1, X^2-1, X^3+X)$ as output basis.

Exercise 2.7

Remind the following theorem. You have to know it, it is furthermore part of the theorems proofs that you have to know:

> **Proposition:**
>
> Let E and F be two finite-dimensional $\mathbb{R}$-vector spaces, $\mathcal{B}_1$ a basis of E, $\mathcal{B}_2$ a basis of F and $f \in \mathcal{L}(E,F)$. Then for all $u \in E$:
>
> $$\mathrm{Mat}_{\mathcal{B}_2}(f(u)) = \mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2}(f) \times \mathrm{Mat}_{\mathcal{B}_1}(u)$$
>
> where $\mathrm{Mat}_{\mathcal{B}_2}(f(u))$ is the column matrix containing the coordinates of $f(u)$ in basis $\mathcal{B}_2$, $\mathrm{Mat}_{\mathcal{B}_1,\mathcal{B}_2}(f)$ the matrix of f in $\mathcal{B}_1$ as input basis and $\mathcal{B}_2$ as output basis and $\mathrm{Mat}_{\mathcal{B}_1}(u)$ is the column matrix containing the coordinates of u in basis $\mathcal{B}_1$.

Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$.

1. What is the linear map $f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ (specify n and p) defined by A in the standard bases as input and output bases?
2. What is the linear map $g \in \mathcal{L}(\mathbb{R}_n[X], \mathbb{R}_p[X])$ (specify n and p) defined by A in the standard bases as input and output bases?
3. Assume that A is the matrix of the linear map $h \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ in $(\varepsilon_1{=}(1,1), \varepsilon_2{=}(1,-1))$ as input basis and the standard basis as output basis. What is the image of $u = (2,-3)$ by h?
4. Assume that A is the matrix of the linear map $\phi \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}_2[X])$ in $(\varepsilon_1{=}(1,2), \varepsilon_2{=}(3,4))$ as input basis and $(P_1{=}1, P_2{=}X+1, P_3 = (X-1)^2)$ as output basis. Let $u = (4,6)$.

   Find the real numbers a, b and c such that $f(u) = aX^2 + bX + c$.

Exercise 2.8

Consider the linear maps:

$$f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (-2x+4y, x-y) \end{cases} \quad \text{and} \quad g : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (x,y,z) & \longmapsto & (z+x, 2z+3x-y) \end{cases}$$

We denote respectively by M_f and M_g the matrices of f and g in the input and output standard bases.

1. Find M_f and M_g. Does the product $M_f \times M_g$ exist? And $M_g \times M_f$? In case of positive answer, compute the product.
2. Does the composition $f \circ g$ exist? And $g \circ f$? In case of positive answer, find the expression of the mapping and find its matrix M in the standard bases.
3. How to get this matrix M from M_f and M_g?

Exercise 2.9

1. Let E be a $\mathbb{R}$-vector space of dimension $n \in \mathbb{N}^*$ and id_E the identity mapping of E. Let $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ be a basis of E.
   (a) What is the matrix I_n of id_E in basis $\mathcal{B}$ as input and output basis?
   (b) Let f be an endomorphism of E and consider its matrix A in basis $\mathcal{B}$ as input and output basis. For all $u \in E$, find $f \circ \mathrm{id}_E(u)$ and $\mathrm{id}_E \circ f(u)$.
   Finally, what are the mappings $f \circ \mathrm{id}_E$ and $\mathrm{id}_E \circ f$? What can you deduce about A and I_n?

2. Let $f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$, $g \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ and $h \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ be the linear maps defined by the following matrices in the standard bases:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$$

   (a) Which of these compositions are possible: $f \circ f$, $f \circ g$, $g \circ f$, $g \circ g$, $g \circ h$, $h \circ g$, $h \circ f$, $f \circ h$, $h \circ h$?

   (b) Compute the matrices of these mappings in the standard bases.

Exercise 2.10

Definition:

- *Let f be a linear map from $\mathbb{R}$-vector space E to a $\mathbb{R}$-vector space F. The **rank** of f, denoted by $Rk(f)$ is the dimension of the image of f: $Rk(f) = dim(Im(f))$.*

- *Furthermore, for a matrix $A \in \mathcal{M}_{n,p}(\mathbb{R})$, the **rank** of A, denoted by $Rk(A)$ is the dimension of the linear subspace spanned by its columns.*

In the following, $0_{n,1}$ will denote $0_{\mathcal{M}_{n,1}(\mathbb{R})}$ that is, the column matrix with n zeros.

1. Let E and F be two finite-dimensional vector spaces, $\mathcal{B} = (e_1, \ldots, e_p)$ a basis of E, $\mathcal{B}'$ a basis of F and $f \in \mathcal{L}(E, F)$. Let $A \in \mathcal{M}_{n,p}(\mathbb{R})$ be the matrix of f in $\mathcal{B}$ as input basis and $\mathcal{B}'$ as output basis.

   (a) Show that $\mathrm{Im}(f) = \mathrm{Span}(f(e_1), \ldots, f(e_p))$.

   (b) Let $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$. Show that

$$\alpha_1 f(e_1) + \ldots + \alpha_p f(e_p) = 0_F \iff \alpha_1 \mathrm{Mat}_{\mathcal{B}'}(f(e_1)) + \ldots + \alpha_n \mathrm{Mat}_{\mathcal{B}'}(f(e_p)) = 0_{n,1}$$

   (c) Deduce that $\mathrm{Rk}(f) = \mathrm{Rk}(A)$.

   (d) Assume that there exists $(\alpha_1, \ldots, \alpha_p) \in \mathbb{R}^p$ such that $\alpha_1 C_1 + \ldots + \alpha C_p = 0_{n,1}$. Show that

$$\alpha_1 e_1 + \ldots + \alpha_p e_p \in \mathrm{Ker}(f)$$

   ($C_1, \ldots, C_p$ are the columns of matrix A)

2. Let $f \in \mathcal{L}(\mathbb{R}_2[X])$ be defined by the following matrix in the standard basis: $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$.

   Find the rank, the image and the kernel of f.

3. Same questions for the linear maps $g \in \mathcal{L}(\mathbb{R}^2)$, $h \in \mathcal{L}(\mathbb{R}_2[X], \mathbb{R})$ and $i \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}_2[X])$ defined by the matrices B, C and D in the standard bases:

$$B = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

★ 4. Same questions for the linear map $\ell \in \mathcal{L}(\mathbb{R}^2)$ whose matrix in $((1,1),(1,-1))$ as input and output basis is:

$$\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}.$$

★ **Exercise 2.11**

We consider the linear maps f and g whose matrices in the standard basis are:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

1. What is the matrix of the linear map $f \circ g$ in the standard basis? That of $g \circ f$?
2. Compute the matrix in the standard basis of the composed of g with itself 10 times, that is, g^{10},.
3. Compute the matrices of f and g in the basis $\big((1,1),(1,-1)\big)$ as input and output bases.
4. Express g^k and f^k for any $k \in \mathbb{N}$.

3 Matri*ces* of a linear map

3.1 Summary

Any linear map f, from a finite dimensional vector space to another, can be associated with a matrix. The latter depending on the bases chosen both for the input and output spaces. Thus, f can be associated with as many different matrices as possible pairs of bases. In this section we will answer the following question : *what are the relations between all these matrices? How do we switch from one to another?* This question is connected with the ability to change the system of *coordinates* we are using, i.e, the bases we are working in.

3.2 Available MiMos

Mimo: Définition de l'inverse d'une matrice

Mimo: Calcul de l'inverse d'une matrice

3.3 Exercises

Exercise 3.12

1. Let f be a linear map: $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined for any $(x,y,z) \in \mathbb{R}^3$ as follows

$$f\big((x,y,z)\big) = (x+y-2z,\, x-y+z,\, -2x+y-z)$$

   (a) Prove that f is bijective and compute f^{-1}.

   (b) Let A be the matrix of f and B the matrix of f^{-1} in the standard basis. Compute AB and BA. The matrix B is denoted by A^{-1}.

2. Accept without proof that the following matrices are invertible. Find their inverses.

$$C = \begin{pmatrix} 2 & 2 & 3 \\ 4 & 3 & 6 \\ -1 & -2 & -2 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 2 & 2 & 3 \\ -2 & 1 & 2 \\ -3 & 2 & 4 \end{pmatrix}$$

Exercise 3.13

Let E be a $\mathbb{R}$-vector space of finite dimension n, $\mathcal{B}_1$ and $\mathcal{B}_2$ two bases of E.

The **transition matrix** from $\mathcal{B}_1$ to $\mathcal{B}_2$ is the square matrix of size $n \times n$, whose columns are the coordinates in $\mathcal{B}_1$ of the $\mathcal{B}_2$'s vectors. It is denoted by $P_{\mathcal{B}_1,\mathcal{B}_2}$.

1. Explain why the transition matrix $P_{\mathcal{B}_1,\mathcal{B}_2}$ is invertible. Interpret the matrix $P^{-1}_{\mathcal{B}_1,\mathcal{B}_2}$.
2. In $\mathbb{R}^3$, consider the standard basis $\mathcal{B}_1$ and a second basis:

$$\mathcal{B}_2 = (\varepsilon_1{=}(1,0,1), \varepsilon_2{=}(0,1,0), \varepsilon_3{=}(1,1,0))$$

   (a) Find the transition matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$. Let P denote this matrix.

   (b) Let $u = (x,y,z) \in \mathbb{R}^3$. Consider the columns matrices $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$, containing the coordinates of vector u in bases $\mathcal{B}_1$ and $\mathcal{B}_2$.
   Express X as a function of the matrices P and X'. Then find an expression of X' as a function of P and X.

   (c) Let $u = (1,2,3) \in \mathbb{R}^3$. Find X and X'.

3. Can these formula be extended to other finite-dimensional vector spaces?

★ **Exercise 3.14**

Back to the linear map from exercise 2-6 (question 1):

$$f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & (2x+y, x-y) \end{cases}$$

Let $\mathcal{B} = (e_1, e_2)$ be the standard basis of $\mathbb{R}^2$ and $\mathcal{B}' = (\varepsilon_1{=}(1,1), \varepsilon_2{=}(1,-1))$ an other basis.

Consider the following matrices (see exercise 2-6, questions 1.b and 1.c):

- A: matrix of f in $\mathcal{B}$ as input and output bases (2-6 (1.a))
- B: matrix of f in $\mathcal{B}'$ as input basis and $\mathcal{B}$ as output basis (2-6 (1.c))
- C: matrix of f in $\mathcal{B}$ as input basis and $\mathcal{B}'$ as output basis (2-6 (1.b))
- A': matrix of f in $\mathcal{B}'$ as input and output bases

In the next questions, our purpose is to find, using matrix A, the matrices B, C and A' of f that we had found at exercise 2-6.

1. What is the transition matrix P from $\mathcal{B}$ to $\mathcal{B}'$?
2. For $u \in \mathbb{R}^2$, consider the columns X and X' containing the coordinates of u in bases $\mathcal{B}$ and $\mathcal{B}'$, then the columns Y and Y' containing the coordinates of $f(u)$ in bases $\mathcal{B}$ and $\mathcal{B}'$.

   What is the matrix relation between X and X'? Between Y and Y'?

   Find the relations between X (or X') and Y (or Y'), involving matrices A, B, C and A'
3. Deduce that $B = AP$ and that $C = P^{-1}A$.
4. Let A' be the matrix f in $\mathcal{B}'$ as input and output basis. Using prevous questions, show that $A' = P^{-1}AP$.